Name: Key

Instructor: ______________________

Time of your class: __________

UMKC Department of Mathematics and Statistics

Math 210 CALCULUS I

Common Final Examination

Saturday, May 7, 2016

INSTRUCTIONS:

1. Write your name, the name of your instructor and the time your class meets on top.
2. Only scientific calculators with no integration capabilities may be used for this examination. Graphing calculators and electronic communication devices are NOT permitted in the room.
3. Make sure that your test contains all problems indicated below on this cover page.
4. On page 1 ONLY: Multiple choice problems; full credit will be given for a correct answer and no credit for an incorrect answer; no work needs to be shown.
5. On all other pages: **All work MUST be shown to receive full credit.**
6. Unless specifically asked not to simplify, all answers need to be simplified completely.
7. EXACT answers are expected, NO approximations (unless otherwise specified).
8. Write your final answer in the box provided.

Problem	Possible	Earned
1	20	
2	6	
3	8	
4	8	
5	12	
6	10	
7	9	
8	9	
9	7	
10	8	
11	9	
12	10	
13	10	
14	10	
15	9	
16	8	
17	9	
18	9	
19	10	
20	10	
21	9	
Total	200	

1. (20 pts) Determine the following limits. Choose "DNE" (Does Not Exist) as your answer only if your function has NO limit, finite or infinite. **Circle one of the indicated choices**. No work needs to be shown.

(a) (5 pts) $\lim_{x\to\infty} \frac{1+x-2x^3}{3x^2+4} = \lim_{x\to\infty} \frac{-2x^3}{3x^2} =$

$= \lim_{x\to\infty} -\frac{2}{3}x = -\infty$

A. DNE

B. ∞

C. $-\infty$

D. $\frac{1}{3}$

E. $\frac{1}{4}$

F. 0

(b) (5 pts) $\lim_{x\to 1^+} \frac{x-3}{x-1} = \frac{-2}{0+}$

A. 1

B. -2

C. -1

D. DNE

E. ∞

F. $-\infty$

(c) (5 pts) $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1$

A. 0

B. 1

C. -1

D. DNE

E. ∞

F. $-\infty$

(d) (5 pts) $\lim_{x\to 0} \frac{x^3-3x^2}{x^3-2x} = \lim_{x\to 0} \frac{x^2(x-3)}{x(x^2-2)}$

$= \lim_{x\to 0} \frac{x(x-3)}{x^2-2} = \frac{0}{-2} = 0$

A. DNE

B. ∞

C. $-\infty$

D. 1.5

E. 1

F. 0

2. (6 pts) Consider the function $f(x) = \begin{cases} x^2+4x-5 & \text{if } x \geq 0 \\ \dfrac{\sin x}{x} & \text{if } x < 0 \end{cases}$

Find the limits below; write DNE if a limit does not exist.

$\lim\limits_{x\to 0^+} f(x) = -5$	$\lim\limits_{x\to 0^-} f(x) = 1$	$\lim\limits_{x\to 0} f(x) = DNE$

3. (8 pts) Write the equation of the line tangent to the graph of $f(x) = \sqrt[3]{x}$ at the point with x-coordinate $x = 8$.

$$f(x) = x^{1/3}$$

$$f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$$

$$f'(8) = \frac{1}{3(\sqrt[3]{8})^2} = \frac{1}{12}$$

$$y - y_0 = m(x - x_0)$$

$$y - 2 = \frac{1}{12}(x-8)$$

$$y = \frac{1}{12}x - \frac{\cancel{8}^{2}}{\cancel{12}_{3}} + 2 = \frac{1}{12}x + \frac{-2+6}{3}$$

$$\boxed{y = \frac{1}{12}x + \frac{4}{3}}$$

4. (8 pts) Use the limit DEFINITION of the derivative to show that the derivative of the function $f(x) = x^2 - x$ is $f'(x) = 2x - 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h-1)}{\cancel{h}} = \lim_{h\to 0} 2x + h - 1$$

$$= 2x - 1$$

5. Compute the derivatives of the following functions. Do not simplify your answers.

(a) (6 pts) $f(x) = \ln(x^2+3)$

$$f'(x) = \frac{1}{x^2+3}(2x)$$

(b) (6 pts) $g(x) = \sqrt{e^{2x}+1}$

$$g'(x) = \frac{1}{2\sqrt{e^{2x+1}}} \cdot e^{2x} \cdot 2$$

(c) (6 pts) $h(x) = x^7 \tan(x)$

$$h'(x) = 7x^6 \tan(x) + x^7 \sec^2(x)$$

6. (10 pts) Let $f(x) = \dfrac{x}{x+2}$. Compute $f'(1)$ and $f''(1)$.

$$f(x) = \frac{x}{x+2}$$

$$f'(x) = \frac{1\cdot(x+2) - 1\cdot x}{(x+2)^2} = \frac{2}{(x+2)^2} = 2\cdot(x+2)^{-2}$$

$$f''(x) = 2(-2)(x+2)^{-3} = -4(x+2)^{-3}$$

$$f'(1) = \frac{2}{(1+2)^2} = \frac{2}{9}$$

$$f''(1) = \frac{-4}{(1+2)^3} = \frac{-4}{27}$$

$$f'(1) = \frac{2}{9}$$

$$f''(1) = -\frac{4}{27}$$

7. (9 pts) Use implicit differentiation to find the derivative $\frac{dy}{dx}$, given that

$$\ln(y) = xy - x + 3$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot y + x\frac{dy}{dx} - 1$$

$$\frac{dy}{dx}\left(\frac{1}{y} - x\right) = y - 1$$

$$\frac{dy}{dx} = \frac{y-1}{\frac{1}{y} - x} = \frac{y-1}{\frac{1-xy}{y}}$$

$$= \frac{y(y-1)}{1-xy}$$

$$\boxed{\frac{dy}{dx} = \frac{y(y-1)}{1-xy}}$$

8. (9 pts) Find the limit and provide a complete justification:

$$\lim_{x\to 0} \frac{\sin(x) - x}{e^x - x - 1} = \lim_{x\to 0} \frac{\cos(x) - 1}{e^x - 1} = \lim_{x\to 0} \frac{-\sin(x)}{e^x} = \frac{0}{e^0} = \frac{0}{1} = 0$$

L'H $\frac{0}{0}$ L'H $\frac{0}{0}$

$$\boxed{0}$$

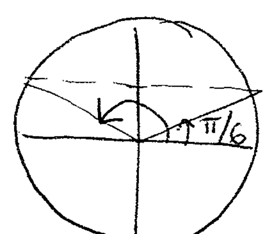
9. (7 pts) A particle is moving along a coordinate line with position function given by $s(t) = 2\cos(t) + t$, with $0 \le t \le 2\pi$. Find all values of the time t at which the particle stopped.

$$v(t) = 2(-\sin(t)) + 1$$

$$v(t) = 0 \rightsquigarrow -2\sin(t) + 1 = 0$$

$$\sin(t) = \frac{1}{2}$$

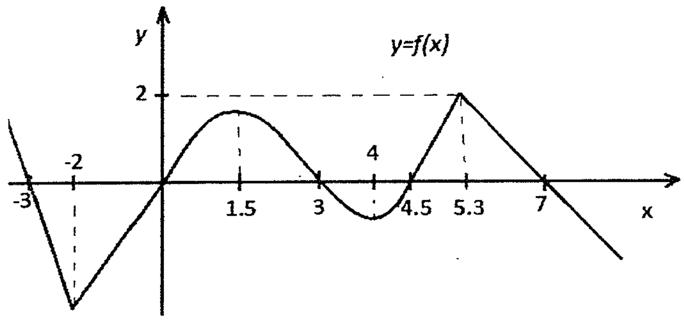
$$t = \frac{\pi}{6}, \frac{5\pi}{6}$$



$$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\boxed{t = \pi/6, \frac{5\pi}{6}}$$

10. (8 pts) The graph of a function is given below.



(a) List the x-coordinates of all points where the derivative $f'(x)$ is equal to zero.

slope of tangent $= 0$

$x = 1.5\ ,\ 4$

(b) List the x-coordinates of all points where the derivative $f'(x)$ does not exist (that is, the function is not differentiable).

no tangent line

$x = -2,\ 5.3$

(c) Approximate the value of $f'(5)$ using the information in the graph.

$$\text{slope} = \frac{2-0}{5.3-4.5} = \frac{2}{0.8} = 2\cdot\frac{10}{8} = \frac{5}{2} = 2.5$$

$f'(5) \simeq 2.5$

11. (9 pts) Find the absolute minimum and the absolute maximum value of $f(x) = 2x^3 - 6x^2 + 5$ over the interval $[-1, 4]$.

$$f'(x) = 6x^2 - 12x = 6x(x-2)$$

$$f'(x) = 0 \rightarrow x = 0,\ x = 2.$$

$$f(-1) = 2(-1)^3 - 6(-1)^2 + 5 = -2-6+5 = -3 \text{ min}$$

$$f(0) = 5$$

$$f(2) = 2\cdot 2^3 - 6\cdot 2^2 + 5 = 2^3(2-3) + 5 = -8+5 = -3 \text{ min}$$

$$f(4) = 2\cdot 4^3 - 6\cdot 4^2 + 5 = 4^2(8-6) + 5 = 32+5 = 37 \text{ max}$$

absolute minimum value $= -3$ at $x = -1, 2$

absolute maximum value $= 37$ at $x = 4$

12. (10 pts) Find the open intervals where the function $f(x) = x^3 \cdot e^x$ is increasing/decreasing, and find the x-coordinates of the points of relative (local) maximum/minimum, if any.

$$f'(x) = 3x^2e^x + x^3e^x = e^x x^2(x+3)$$

$$f'(x) = 0 \rightsquigarrow x = 0, x = -3.$$

$$\begin{array}{ccccc} - & & + & & + \\ & -3 & & 0 & \\ \searrow & & \nearrow & & \nearrow \end{array}$$

$f(x)$ is increasing on:	$(-3, \infty)$
$f(x)$ is decreasing on:	$(-\infty, -3)$
relative(local) minimum(s) at $x =$	-3
relative(local) maximum(s) at $x =$	NONE

13. (10 pts) Given the function $f(x) = x^4 - 12x^2$, find the open intervals on which f is concave up.

$$f'(x) = 4x^3 - 24x$$

$$f''(x) = 12x^2 - 24 = 12(x^2 - 2)$$

$$f''(x) = 0 \rightsquigarrow x^2 - 2 = 0$$

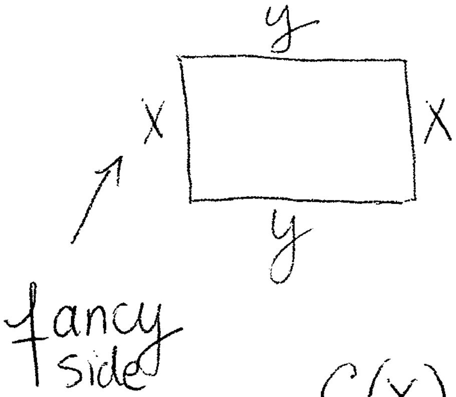
$$x^2 = 2$$

$$x = \pm\sqrt{2}.$$

$$\begin{array}{ccccc} + & & - & & + \\ & -\sqrt{2} & & \sqrt{2}. & \\ \cup & & \cap & & \cup \end{array}$$

concave up on:	$(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

14. (10 pts) A farmer wants to fence a rectangular part of his land, as to enclose an area of 30,000 square feet. The fenced area is to have one border shared with a neighbor which he wishes to be fancy so he will spend 25 dollars per foot on that side and he will use only 5 dollars per foot on fencing the remaining sides. What should be the length of the fancy side so that the cost of the fence is minimized?



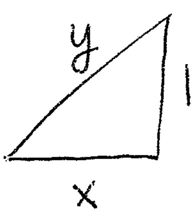
$$\text{Cost} = 25\cdot x + y\cdot 5 + y\cdot 5 + x\cdot 5$$
$$= 30x + 10y$$
$$x\cdot y = 30{,}000 \rightsquigarrow y = \frac{30{,}000}{x}.$$
$$C(x) = 30x + 10\cdot\frac{30{,}000}{x}$$
$$C'(x) = 30 + 300{,}000\cdot\left(-\frac{1}{x^2}\right)$$
$$C'(x) = 0 \rightsquigarrow \cancel{30}^{\,1} = \cancel{300{,}000}^{\,10{,}000}\cdot\frac{1}{x^2}$$
$$x^2 = 10{,}000$$
$$x = 100$$

$-$ $+$

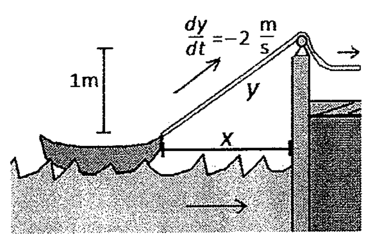
100

$$\boxed{100}$$

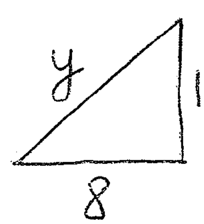
15. (9 pts) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the deck that is 1 m higher than the bow of the boat. If the rope is pulled at a rate of 2 m/s, how fast is the boat approaching the boat at the instant when it is 8 m away from the dock ?



$$y^2 = x^2 + 1^2$$
$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$



$$x = 8,\ y = \sqrt{67},\ \frac{dy}{dt} = -2$$



When $x = 8$:
$$y = \sqrt{8^2+1}$$
$$= \sqrt{67}$$

$$\cancel{2}\cdot\sqrt{67}\,(-2) = \cancel{2}\cdot 8\cdot\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{-\sqrt{67}}{4}\ m/s$$

$$\boxed{\left.\frac{dx}{dt}\right|_{x=8} = -\frac{\sqrt{67}}{4}\ m/s}$$

16. (8 pts) A particle moves along an s-axis with velocity $v(t)$. If $v(t) = 5\sin(t)$ and $s(\pi) = 2$, find the position function $s(t)$.

$$s(t) = \int 5\sin(t)\,dt = -5\cos t + C$$
$$s(\pi) = -5\cos\pi + C$$
$$+2 = -5(-1) + C$$
$$+2 = 5 + C \rightarrow C = -3$$

$s(t) = \boxed{-5\cos t - 3}$

17. (9 pts) Use a substitution to compute the integral $I = \int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx$. Show all work.

$u = x^3 + 1$
$du = 3x^2dx \rightsquigarrow x^2dx = \frac{1}{3}du$

$x = 0 \rightarrow u = 1$
$x = 2 \rightarrow u = 9$

$$I = \int_1^9 \frac{1}{\sqrt{u}} \cdot \frac{1}{3}\,du = \frac{1}{3}\int_1^9 u^{-1/2}\,du$$
$$= \frac{1}{3}\left.\frac{u^{1/2}}{1/2}\right|_1^9 = \frac{2}{3}\sqrt{u}\Big|_1^9 = \frac{2}{3}\left(\sqrt{9} - \sqrt{1}\right)$$
$$= \frac{2}{3}(3-1) = \frac{4}{3}$$

$I = \boxed{\frac{4}{3}}$

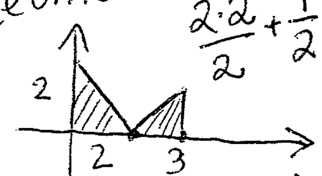
18. (9 pts) Compute the integral $I = \int_0^3 |x-2|\,dx$. (Do not ignore the absolute value.)

$x-2$

$-$ | $+$ (sign chart, split at 2)

$$\int_0^2 \underbrace{-(x-2)}_{-x+2}\,dx + \int_2^3 x - 2\,dx$$
$$= -\frac{x^2}{2} + 2x\Big|_0^2 + \frac{x^2}{2} - 2x\Big|_2^3$$
$$= -\frac{2^2}{2} + 2\cdot 2 - 0 + \frac{3^2}{2} - 2\cdot 3 - \left(\frac{2^2}{2} - 2\cdot 2\right)$$
$$= \cancel{-2} \cancel{+4} + \frac{9}{2} - 6 \cancel{-2} + 4 = \frac{9}{2} - 2 = \frac{5}{2}$$

geometric solution
$$\frac{2\cdot 2}{2} + \frac{1}{2} = \frac{5}{2}$$



OK

$I = \boxed{\frac{5}{2}}$

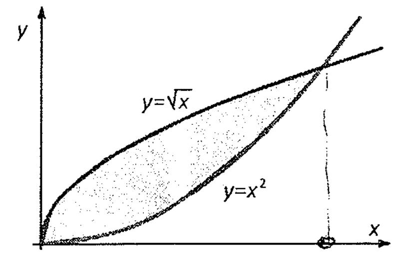
19. (10 pts) Evaluate the indefinite integral $\int \frac{3}{2(1+x)} + \frac{1}{5x^2} + \frac{1}{x^2+1}\,dx$.

$$= \frac{3}{2}\int \frac{1}{1+x}\,dx + \frac{1}{5}\int x^{-2}\,dx + \int \frac{1}{x^2+1}\,dx = \frac{3}{2}\ln|x+1| + \frac{1}{5}\frac{x^{-1}}{-1} + \tan^{-1}(x)$$

$$\boxed{\frac{3}{2}\ln|x+1| - \frac{1}{5}\frac{1}{x} + \tan^{-1}(x) + C}$$

20. (10 pts) Find the area in the first quadrant enclosed between $y = \sqrt{x}$ and $y = x^2$.

$$\int_0^1 \underbrace{\sqrt{x}}_{x^{1/2}} - x^2\,dx = \frac{x^{3/2}}{3/2} - \frac{x^3}{3}\Big|_0^1$$



$$= \frac{2}{3}\left(\sqrt{x}\right)^3 - \frac{x^3}{3}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$\sqrt{x} = x^2$ 1

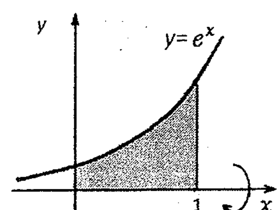
$x = x^4$

$x(x^3-1) = 0 \rightarrow x = 0, 1$

AREA= $\frac{1}{3}$

21. (9 pts) Consider the region under the graph of the function $f(x) = e^x$ over the interval $[0, 1]$. Compute the volume of the solid obtained by rotating this region about the x-axis.

$$\int_0^1 \pi (e^x)^2\,dx = \int_0^1 \pi \cdot e^{2x}\,dx$$



$$= \pi \cdot \frac{e^{2x}}{2}\Big|_0^1 = \frac{\pi}{2}(e^2 - e^0)$$

$$= \frac{\pi}{2}(e^2 - 1)$$

VOLUME= $\frac{\pi}{2}(e^2-1)$