Name: Key

Instructor:

Time of your class:

UMKC Department of Mathematics and Statistics

Math 210 CALCULUS I
Common Final Examination

Saturday, December 10, 2016

INSTRUCTIONS:

1. Write your name, the name of your instructor and the time your class meets on top.
2. Only scientific calculators with no integration capabilities may be used for this examination. Graphing calculators and electronic communication devices are NOT permitted in the room.
3. Make sure that your test contains all problems indicated below on this cover page.
4. On page 1 ONLY: Multiple choice problems; full credit will be given for a correct answer and no credit for an incorrect answer; no work needs to be shown.
5. On all other pages: **All work MUST be shown to receive full credit.**
6. Unless specifically asked not to simplify, all answers need to be simplified completely.
7. EXACT answers are expected, NO approximations (unless otherwise specified).
8. Write your final answer in the box provided.

Problem	Possible	Earned
1	20	
2	5	
3	8	
4	8	
5	20	
6	5	
7	8	
8	6	
9	9	
10	8	
11	12	
12	8	
13	10	
14	10	
15	9	
16	10	
17	6	
18	9	
19	8	
20	12	
21	9	
Total	200	

1. (20 pts) Determine the following limits. Choose "DNE" (Does Not Exist) as your answer only if your function has NO limit, finite or infinite. **Circle one of the indicated choices.** No work needs to be shown.

(a) (5 pts) $\lim\limits_{x\to-\infty} \dfrac{2x^4+5x-1}{3x^3+4} =$

$\frac{2}{3}x(-\infty)$

A. DNE

B. ∞

C. $-\infty$

D. $\frac{2}{3}$

E. $-\frac{1}{4}$

F. 0

(b) (5 pts) $\lim\limits_{x\to1^-} \dfrac{x+2}{x-1} =$

$\frac{+}{-}$

A. 1

B. -2

C. $-\frac{1}{2}$

D. DNE

E. ∞

F. $-\infty$

(c) (5 pts) $\lim\limits_{x\to1} \dfrac{\sin(x-1)}{x^2-1} =$

A. 0

B. 1

C. -1

D. $\frac{1}{2}$

E. 2

F. ∞

(d) (5 pts) $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+5}}{2x^2+1} =$

A. 0

B. ∞

C. 1

D. 2

E. DNE

F. 5

NO PARTIAL CREDIT

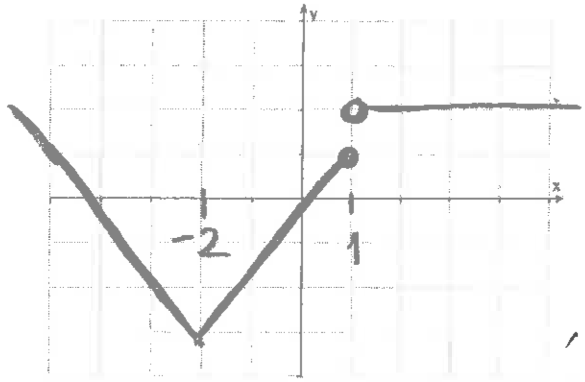
2. (5 pts) Draw below the graph of a function defined on $(-\infty,\infty)$ such that the following conditions are satisfied:

1pt for this

2 pts (a) The limits $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ both exist, but $\lim_{x\to 1} f(x)$ does not exist.

2pts (b) $f(x)$ is continuous but not differentiable at $x=-2$.

other shapes possible



Subtract 2pts if (a) does not hold.

Subtract 2pts if (b) does not hold

3. (8 pts) Let $f(x)=x^3$. Write the equation of the line tangent to the graph $y=f(x)$ at the point with x-coordinate $x=2$.

2pts $f'(x)=3x^2$

1pt $f'(2)=3\cdot 4=12$

$f(2)=2^3=8$

$y-y_0=m(x-x_0)$ or $y=f(a)+f'(a)(x-a)$

$y-8=12(x-2)$

$y=8+12x-24=\underbrace{12x-16}_{1pt}$

4 pts

$y=$ $\boxed{12x-16.}$

4. (8 pts) Use the limit DEFINITION of the derivative to show that the derivative of the function $f(x)=2-3x^2$ is $f'(x)=-6x$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{2-3(x+h)^2-(2-3x^2)}{h}$$

3pts

$$=\lim_{h\to 0}\frac{2-3(x^2+2xh+h^2)-2+3x^2}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{2}-\cancel{3x^2}-6xh-3h^2-\cancel{2}+\cancel{3x^2}}{h}=\lim_{h\to 0}\frac{\cancel{h}(-6x-3h)}{\cancel{h}}$$

3pts

0

$$=-6x$$

2pts

5. Compute the derivatives of the following functions. Do NOT simplify.

(a) (5 pts) $f(x) = \sqrt{x^5 + 2^x}$

alternately: $\frac{1}{2}(x^5+2^x)^{-1/2}$

$f'(x) = \boxed{\dfrac{1}{2\sqrt{x^5+2^x}} \cdot (5x^4 + 2^x \ln 2)}$

(b) (5 pts) $g(x) = \sin(x^2) - \sin^2(x)$

$g'(x) = \boxed{\cos(x^2)\cdot(2x) - 2\sin(x)\cdot\cos(x)}$

(c) (5 pts) $h(x) = \dfrac{e^{3x}}{e^x + 1}$

$= e^{4x}$

$h'(x) = \boxed{\dfrac{e^{3x}\cdot 3\cdot(e^x+1) - e^{3x}\cdot e^x}{(e^x+1)^2}}$

(d) (5 pts) $j(x) = x^3 \tan(x)$

$j'(x) = \boxed{3x^2 \cdot \tan(x) + x^3 \cdot \sec^2(x)}$

6. (5 pts) Use logarithmic diffentiation to compute $\frac{dy}{dx}$ for $y = x^x$. Show all work.

$y = x^x$

$\left.\begin{aligned} \ln y &= \ln(x^x) \\ \ln y &= x \ln(x) \end{aligned}\right\}$ 1pt

$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln(x) + x \cdot \frac{1}{x}$ ← 3pts

$\left.\begin{aligned} \frac{dy}{dx} &= y(\ln x + 1) \\ &= x^x(\ln x + 1) \end{aligned}\right\}$ 1pt

Do not subtract if they write y instead of $\frac{dy}{dx}$

$\frac{dy}{dx} = \boxed{x^x(\ln x + 1)}$

7. (8 pts) Use implicit differentiation to find the derivative $\frac{dy}{dx}$, given that $y^2 - 3xy + x^2 = 1$.

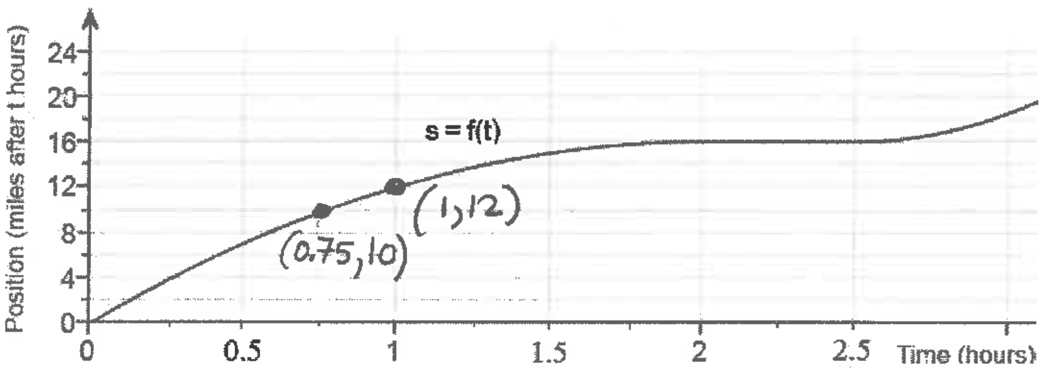
$$y^2 - 3xy + x^2 = 1$$

$$\left.\begin{array}{l} \frac{d}{dx}\left[y^2 - 3xy + x^2\right] = \frac{d}{dx}[1] \\ 2y\frac{dy}{dx} - 3 \cdot 1y - 3x\frac{dy}{dx} + 2x = 0 \end{array}\right\} \text{5 pts}$$

$$\left.\begin{array}{l} \frac{dy}{dx}(2y - 3x) = -2x + 3y \\ \frac{dy}{dx} = \frac{-2x + 3y}{2y - 3x} \end{array}\right\} \text{3 pts}$$

$$\boxed{\frac{dy}{dx} = \frac{-2x + 3y}{2y - 3x}}$$

8. (6 pts) The following graph shows the position s of a bicyclist after t hours from a start time.



(a) (2 pts) Does the bicyclist speed up or slow down during the first two hours? Explain.

Velocity $= f'(t) =$ slope of tangent line.
The slopes seem to be decreasing (and positive) $\longrightarrow$ Slowing down
(Alternate justification: concave down & increasing)

(b) Based on the information in the graph, appoximate as accurately as possible the following velocities. Explain clearly how the answer has been obtained.

other choices OK, if close enough to the tangent

i. (2 pts) The (instantaneous) velocity when $t = 1$;

We will use the points (1,12) and (0.75,10) to approximate the slope of the tangent line.

$$m \approx \frac{12 - 10}{1 - 0.75} = \frac{2}{0.25} = 8$$

$\boxed{8 \text{ m/h}}$

give credit for any answer between 6 and 10 if the work is correct

ii. (2 pts) The (instantaneous) velocity when $t = 2.2$.

At $t = 2.2$ the tangent line seems horizontal $\rightsquigarrow$ slope $= 0$.
(Alternately: the bicyclist is stationary from $t = 2$ to $t = 2.5$)

$\boxed{0 \text{ m/h}}$

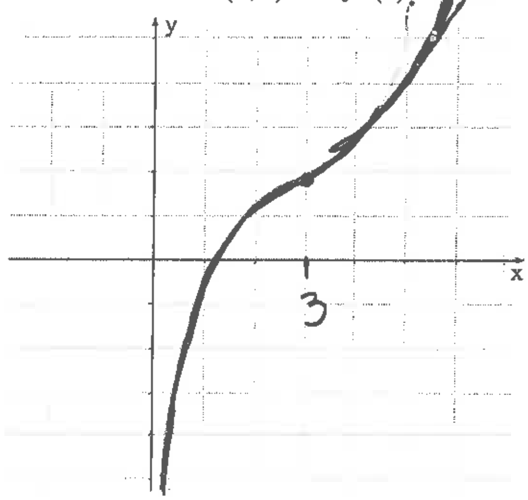
1pt

9. (9 pts) Sketch the graph of a function with the following properties: 2pts

(a) The domain of $f(x)$ is the interval $(0,\infty)$ and $f(x)$ has vertical asymptote at $x=0$.

2pts (b) $f'(x)>0$ on the interval $(0,\infty)$.

4 pts (c) $f''(x)<0$ for all x on the interval $(0,3)$ and $f''(x)>0$ on the interval $(3,\infty)$.



10. (8 pts) Let $f(x)=x+\dfrac{2}{x}$. Find the absolute minimum and the absolute maximum value of $f(x)$ on the interval $[1,3]$.

$f'(x)=1-\dfrac{2}{x^2}$ 1pt

2pt

$f'(x)=0 \rightsquigarrow 1-\dfrac{2}{x^2}=0$

$1=\dfrac{2}{x^2} \rightsquigarrow x^2=2$

$x=\pm\sqrt{2}$

$x=\sqrt{2}$ C.P.

1pt $f(1)=1+\frac{2}{1}=3$

1pt $f(\sqrt{2})=\sqrt{2}+\dfrac{2}{\sqrt{2}}=2\sqrt{2}\approx 2.82$

1pt $f(3)=3+\frac{2}{3}=\frac{11}{3}\approx 3.66$

1pt absolute minimum value = $2\sqrt{2}$ at $x=\sqrt{2}$

1pt absolute maximum value = $\frac{11}{3}$ at $x=3$

11. (12 pts) Use L'Hôpital's Rule to compute the limits below. Show all work.

(a) (6 pts) $\displaystyle\lim_{x\to3}\frac{e^x-e^3}{\sin(x-3)} \overset{L'H}{\underset{\frac{0}{0}}{=}} \underbrace{\lim_{x\to3}\frac{e^x}{\cos(x-3)}}_{4pts}=\underbrace{\frac{e^3}{\cos(0)}=e^3}_{2pts}$

$\boxed{e^3}$

(b) (6 pts) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \underbrace{\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}}}_{4pts}=\underbrace{\lim_{x\to\infty}\frac{1}{2\sqrt{x}}\cdot\frac{x}{1}=\lim_{x\to\infty}\frac{\sqrt{x}}{2}=\infty}_{2pts}$

$\boxed{\infty}$

12. (8 pts) The first derivative of a function $f(x)$ has been computed to be $f'(x) = \dfrac{(x-2)^2(x+1)}{x-3}$. Find the open intervals where $f(x)$ is increasing/decreasing and find the x-coordinates of the points of local (relative) maximum/minimum, if any.

$f'(x)=0 \to x=2, x=-1$

$f'(x)$ undefined $\to x=3$

+2pts for some work

x		-1		2		3	
sign of f'	$+$		$-$		$-$		$+$
f	↗		↘		↘		↗

Give at most 4 pts partial credit for an incorrect sign chart followed by correct interpretation.

3pts $f(x)$ is increasing on: $(-\infty,-1),(3,\infty)$

1pt $f(x)$ is decreasing on: $(-1,2),(2,3)$

OK if they write $(-1,3)$

1pt point(s) of local (relative) minimum at $x=$ ~~3~~ → NA is OK because we do not know if $f(x)$ is defined at $x=3$

1pt point(s) of local (relative) maximum at $x= -1$

13. (10 pts) Given the function $f(x) = \frac{1}{6}x^6 - 5x^4$, find the open intervals on which f is concave up/concave down and the inflection point(s).

1pt $f'(x) = \frac{1}{6}\cdot 6x^5 - 5\cdot 4x^3 = x^5 - 20x^3$

1pt $f''(x) = 5x^4 - 20\cdot 3x^2 = 5x^4 - 60x^2 = 5x^2(x^2-12)$

3pts $f''(x)=0 \to x=0, x=\pm\sqrt{12}$

x		$-\sqrt{12}$		0		$\sqrt{12}$	
sign of f''	$+$		$-$		$-$		$+$
f	∪		∩		∩		∪

concave down on: $(-\sqrt{12},0),(0,\sqrt{12})$ 2pts

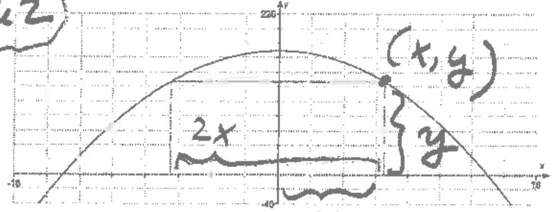
$(-\sqrt{12},\sqrt{12})$ OK

concave up on: $(-\infty,-\sqrt{12}),(\sqrt{12},\infty)$ 2pts

inflection point(s) at $x= -\sqrt{12}, \sqrt{12}$ 1pt

Subtract 1pt $\boxed{\text{at the end}}$ for the last item in this case

14. (10 pts) A rectangle is constructed with its base on the x-axis and two of its vertices on the parabola $y = 169 - x^2$. What are the dimensions (length and width) of the rectangle with the maximum area? Find the maximum area.



give this point and all others if they forget the 2

1pt $\rightarrow$ Area $= 2xy$

2pts $A(x) = 2x(169 - x^2) = 2\cdot 169x - 2x^3$

$x > 0$

2pts $\left\{ \begin{aligned} A'(x) &= 2\cdot 169 - 6x^2 \\ &= 2(169 - 3x^2) \quad \text{or} \quad 338 - 6x^2 \end{aligned} \right.$

$\left. \begin{aligned} A'(x) = 0 \rightsquigarrow 3x^2 &= 169 \\ x^2 &= \frac{169}{3} \end{aligned} \right\}$ 2pts

check max:

$+ \quad -$

$0 \nearrow \frac{13}{\sqrt{3}} \searrow$

Subtract 1pt one time if approximations are given instead $\rightarrow$

$x = \sqrt{\frac{169}{3}} = \frac{13}{\sqrt{3}}$ OR $\frac{13\sqrt{3}}{3}$

$y = 169 - x^2 = 169 - \frac{169}{3} = \frac{2\cdot 169}{3}$

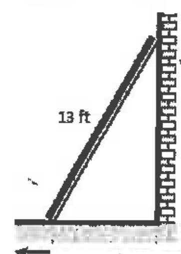
Area $= 2xy = 2\cdot\frac{13}{\sqrt{3}}\cdot\frac{2\cdot 169}{3} = \frac{8788}{3\sqrt{3}}$

1pt length $= \frac{26}{\sqrt{3}}$ or $\frac{26\sqrt{3}}{3}$

1pt width $= \frac{338}{3}$

1pt maximum area $= \frac{8788}{3\sqrt{3}}$ or $\frac{8788\sqrt{3}}{9}$

15. (9 pts) A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot of the ladder be moving away from the wall when the top is 5 ft above the ground?



Know: $\frac{dy}{dt} = -2$ ft/s

Want: $\frac{dx}{dt} = ?$ when $y = 5$ ft

pay attention to notation (some students may have the x and y switched...)

1pt for labeling the picture and translating info.

1pt $x^2 + y^2 = 13^2$

1pt $y = 5 \rightsquigarrow x^2 = 13^2 - 5^2$

$x^2 = 169 - 25 = 144$

$x = 12$

3pts $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

3pts $\left\{ \begin{aligned} &2\cdot 12\frac{dx}{dt} + 2\cdot 5(-2) = 0 \\ &\frac{dx}{dt} = \frac{\cancel{2}\cdot 5(\cancel{+2})}{\cancel{2}\cdot \cancel{12}\,6} = \frac{5}{6} \end{aligned} \right.$

$\boxed{\frac{5}{6}\text{ ft/s}}$

16. (10 pts) Compute the integral: $I=\int \frac{x^3+1}{x}+\frac{1}{x^2+1}+5\sqrt{x}\,dx$

$$\int \frac{x^3}{x}+\frac{1}{x}+\frac{1}{x^2+1}+5x^{1/2}dx = \frac{x^3}{3}+\ln|x|+\tan^{-1}x+5\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+C$$

x^2

$5\cdot\frac{2}{3}=\frac{10}{3}$

2pts 2pts

$$\boxed{I=\frac{x^3}{3}+\ln|x|+\tan^{-1}x+\frac{10}{3}x^{\frac{3}{2}}+C} \leftarrow \text{1pt}$$

2pts 1pt 2pts

17. (6 pts) Compute the integral $I=\int_0^{\frac{\pi}{6}}\sin(x)\,dx$

$$\underbrace{-\cos x\Big|_0^{\frac{\pi}{6}}}_{\text{2pts}} = \underbrace{-\cos\frac{\pi}{6}-(-\cos 0)}_{\text{2pts}} = \underbrace{-\frac{\sqrt{3}}{2}+1}_{\text{2pts}}$$

$$\boxed{I=-\frac{\sqrt{3}}{2}+1}$$

18. (9 pts) Use a change of variable (u-substitution) to compute the integral $I=\int\frac{x^2}{(5x^3+1)^6}dx.$

1pt $u=5x^3+1$

1pt $du=15x^2dx \rightsquigarrow x^2dx=\frac{1}{15}du$

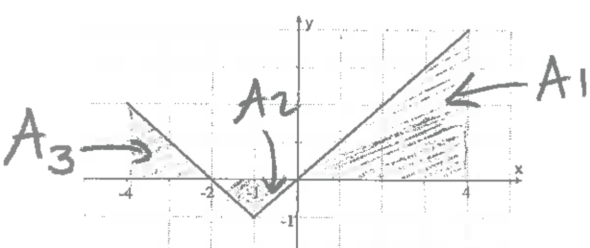
4pts

$$\int\frac{x^2}{(5x^3+1)^6}dx = \boxed{\int\frac{1}{\underbrace{u^6}_{u^{-6}}}\cdot\frac{1}{15}du} = \underbrace{\frac{1}{15}\cdot\frac{u^{-5}}{-5}}_{\text{2pts}}+C = -\frac{u^{-5}}{75}+C$$

$$=-\frac{1}{75}(5x^3+1)^{-5}+C \leftarrow \text{1pt}$$

$$\boxed{I=-\frac{1}{75}(5x^3+1)^{-5}+C}$$

19. (8 pts) Consider the function $f(x)$ whose graph is given below. Use geometry to compute the following integrals:



$A_1=\frac{4\cdot4}{2}=8$

$A_2=\frac{1\cdot2}{2}=1$

$A_3=\frac{2\cdot2}{2}=\frac{4}{2}=2$

$8-1+2$

$\int_0^4 f(x)dx=$ 8	$\int_{-2}^0 f(x)dx=$ -1	$\int_{-4}^{-2} f(x)dx=$ 2	$\int_{-4}^4 f(x)dx=$ 9
2pts	2pts	2pts	2pts

$\leftarrow$ give 1pt if the answer is positive

20. (12 pts) The velocity of an object moving along a line is given by $v(t) = 3t^2 - 12$, for $0 \le t \le 5$.

(a) (6 pts) Find the displacement of the object during the given time interval.

$$\int_0^5 3t^2 - 12\,dt = \frac{\cancel{3}t^3}{\cancel{3}} - 12t\Big/_0^5 = 5^3 - 12\cdot 5$$
$$= 125 - 60 = 65$$

displacement = 65

(b) (6 pts) Find the distance travelled during the given time interval.

$v(t) = 0 \rightarrow 3t^2 - 12 = 0$
$3t^2 = 12 \rightarrow t^2 = 4 \rightarrow t = 2 \quad (t \ge 0)$

$-$ $+$ ← sign chart for velocity

0 2 5

$$\int_0^2 3t^2 - 12\,dt = t^3 - 12t\Big/_0^2 = 8 - 24 = -16 \quad \text{2pts}$$

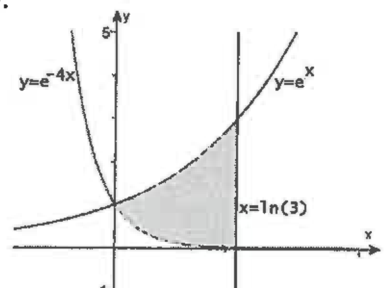
$$\int_2^5 3t^2 - 12\,dt = t^3 - 12t\Big/_2^5 = 65 - (-16) = 65 + 16$$
$$= 81 \quad \text{2pts}$$

2pts

$$|-16| + 81 = 16 + 81$$
$$= 97$$

distance travelled = 97

21. (9 pts) Find the area of the region bounded by $y = e^x$, $y = e^{-4x}$ and $x = \ln(3)$.



$$A = \underbrace{\int_0^{\ln 3} e^x - e^{-4x}\,dx}_{\text{4pts}} = \underbrace{e^x - \frac{e^{-4x}}{-4}\Big/_0^{\ln 3}}_{\text{2pts}}$$

2pts → $= e^{\ln 3} + \dfrac{e^{-4\ln 3}}{4} - \left(e^0 + \dfrac{e^0}{4}\right)$

1pt → $= 3 + \dfrac{\left(\frac{1}{3}\right)^4}{4} - 1 - \dfrac{1}{4} = 2 + \dfrac{1}{324} - \dfrac{1}{4}$

give full credit if they get here

$$\frac{2\cdot 324 + 1 - 81}{324} = \frac{568}{324} = \frac{284}{162}$$

AREA= $\dfrac{142}{81}$