Name: SOLUTIONS/KEY

Instructor: ______________________

Time of your class: ____________

UMKC Department of Mathematics and Statistics

Math 210 CALCULUS I

Common Final Examination

Saturday, May 10, 2014

INSTRUCTIONS:

1. Write your name, the name of your instructor and the time your class meets on top.
2. Only scientific calculators may be used for this examination. Graphing calculators and electronic communication devices are NOT permitted in the room.
3. Make sure that your test contains all problems indicated below on this cover page.
4. On page 1 ONLY: Multiple choice problems; full credit will be given for a correct answer and no credit for an incorrect answer; no work needs to be shown.
5. On all other pages: **All work MUST be shown to receive full credit.**
6. Unless specifically asked not to simplify, all answers need to be simplified completely.
7. EXACT answers are expected, NO approximations (unless otherwise specified).
8. Write your final answer in the box provided.

	Problem	Possible	Earned
grade these when finished →	1	15	
	2	9	
	3	9	
Liana →	4	9	
Megan	5	5	
	6	11	
	7	8	
Bill	8	9	
	9	16	
	10	8	
Justin	11	10	
	12	9	
	13	8	
a	14	9	
	15	9	
	16	9	
	17	8	
	[illegible]	9	
	[illegible]	5	
	[illegible]	6	
	[illegible]	10	
	22	9	
	Total	200	

1. (15 pts) Determine the following limits. Choose "DNE" (Does Not Exist) as your answer only if your function has NO limit, finite or infinite. **Circle one of the indicated choices.** No work needs to be shown.

(a) (5 pts) $\lim_{x\to\infty} \dfrac{-x^4+3x-1}{x^3-2x+1} =$

NO PARTIAL CREDIT ON THIS PAGE

A. 1

B. 0

C. -1

D. DNE

E. ∞

F. $-\infty$

(b) (5 pts) $\lim_{x\to 3^+} \dfrac{x}{x-3} =$ $\dfrac{3}{0^+} = \infty$

A. 0

B. 1

C. 3

D. DNE

E. ∞

F. $-\infty$

(c) (5 pts) $\lim_{y\to 6} \dfrac{y^2-36}{y-6}$ $= \lim_{y\to 6} \dfrac{\cancel{(y-6)}(y+6)}{\cancel{y-6}}$

$=12$

A. 1

B. 6

C. 12

D. 36

E. DNE

F. ∞

2. (9 pts) Consider the function $f(x) = \begin{cases} x-4 & \text{for } x \le 0 \\ x^2+1 & \text{for } x > 0 \end{cases}$

(a) (6 pts) Find the following limits; write "DNE" if the limit is undefined.

2pts each

$0-4$ $\quad$ $\boxed{\lim_{x\to 0^-} f(x) = -4}$

$0+1$ $\quad$ $\boxed{\lim_{x\to 0^+} f(x) = 1}$

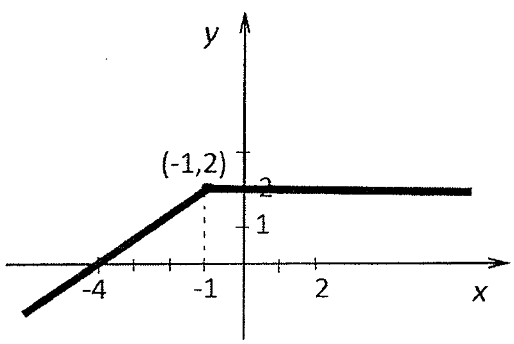
3^2+1 $\quad$ $\boxed{\lim_{x\to 3} f(x) = 10}$

(b) (3 pts) Is the function $f(x)$ above continous for all values of x? Why or why not?

1pt No, because $\lim_{x\to 0} f(x)$ does not exist, since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

2pts for explanation

3. (9 pts) The graph of a function $f(x)$ is sketched below.



Find the following derivatives. If a derivative does not exist, write "DNE".

3pts each

$\boxed{f'(-4) = \frac{2}{3}}$ $\quad$ $\boxed{f'(-1) = \text{DNE}}$ $\quad$ $\boxed{f'(2) = 0}$

4. (9 pts) Find the equation of the line tangent to the graph of $y = x^2$ at the point $(-3, 9)$.

$\frac{dy}{dx} = 2x$ 2pts

slope: $m = 2\cdot(-3) = -6$ 2pts

$y - 9 = -6(x-(-3))$ 3pts

$y = -6(x+3)+9$

$y = -6x - 18 + 9$

$= -6x - 9$ 2pts

$\boxed{y = -6x - 9}$

5. (5 pts) Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

A spherical balloon is being inflated. Find the rate of change of the volume V with respect to the radius r at the instant when $r = 4$.

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2 \quad \text{3pts}$$

$$\left.\frac{dV}{dr}\right|_{r=4} = 4\pi(4^2) = 64\pi \quad \text{2pts}$$

rate of change= 64π

6. (11 pts) Find the derivatives of the following functions. Do NOT simplify your answer.

(a) (5 pts) $g(x) = \dfrac{x^3}{x^2 - 1}$

Subtract 1pt if + instead of -

1pt 1pt 1pt 1pt

$$g'(x) = \frac{3x^2(x^2-1) - x^3(\cancel{x^2-1})(2x)}{(x^2-1)^2}$$

1pt

(b) (6 pts) $h(x) = e^{3x}\ln(x^2+1)$

$$h'(x) = 3 \cdot e^{3x}\ln(x^2+1) + e^{3x} \cdot \frac{1}{x^2+1} \cdot 2x$$

1pt 1pt 1pt 1pt 1pt 1pt

7. (8 pts) Let $f(x) = \sqrt{3x+1}$. Compute $f'(1)$ and $f''(1)$.

$$f(x) = (3x+1)^{1/2}$$

$$f'(x) = \frac{1}{2}(3x+1)^{-1/2} \cdot 3 = \frac{3}{2}(3x+1)^{-1/2} \quad \text{3pts}$$

$$f''(x) = \frac{3}{2}\cdot\left(-\frac{1}{2}\right)(3x+1)^{-3/2} \cdot 3 = -\frac{9}{4}(3x+1)^{-3/2} \quad \text{3pts}$$

$$f'(1) = \frac{3}{2\cdot\sqrt{4}} = \frac{3}{4} \quad \text{1pt}$$

$$f''(1) = -\frac{9}{4(\sqrt{4})^3} = -\frac{9}{4\cdot 8} = -\frac{9}{32} \quad \text{1pt}$$

$f'(1) = \frac{3}{4}$

$f''(1) = -\frac{9}{32}$

8. (9 pts) Find $\frac{dy}{dx}$ by differentiating implicitly: $x^6 - 5x^2y^2 + y^3 = 6$.

$$6x^5 - 10xy^2 - 5x^2 \cdot 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$ (6pts)

$$\frac{dy}{dx}\left(3y^2 - 10x^2y\right) = 10xy^2 - 6x^5$$ (2pts)

(1pt)

$$\boxed{\frac{dy}{dx} = \frac{10xy^2 - 6x^5}{3y^2 - 10x^2y}}$$

9. (16 pts) Find the limits and justify completely the answers.

(a) (8 pts) $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} \underset{L'H}{\overset{0/0}{=}} \lim_{x\to 1} \frac{\cos(\pi x)\cdot \pi}{\frac{1}{x}}$ (4pts)

(for the attempt to use L'H) (2pts)

$$= \lim_{x\to 1} x\cdot\cos(\pi x)\cdot\pi = 1\cdot\cos(\pi)\cdot\pi = -\pi$$

(2pts)

$$\boxed{-\pi}$$

(b) (8 pts) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \underset{L'H}{\overset{0/0}{=}} \lim_{x\to 0} \frac{e^x - 1}{2x}$ (2pts)

(1pts)

$$\underset{L'H}{\overset{0/0}{=}} \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

(1pts) (2pts) (2pts)

$$\boxed{\frac{1}{2}}$$

10. (8 pts) Consider the function $f(x) = x^2 - 3x$. Show, using the **DEFINITION** of the derivative, that $f'(x) = 2x - 3$.

3pts for knowing the def.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} = \lim_{h \to 0} 2x + h - 3 = 2x - 3$$

5pts

11. (10 pts) The following information about a function $f(x)$ is known:

2 (a) $f(x)$ is defined and differentiable for all real x, except for $x = 2$.

2 (b) $\lim_{x \to -\infty} f(x) = 0 = \lim_{x \to \infty} f(x)$.

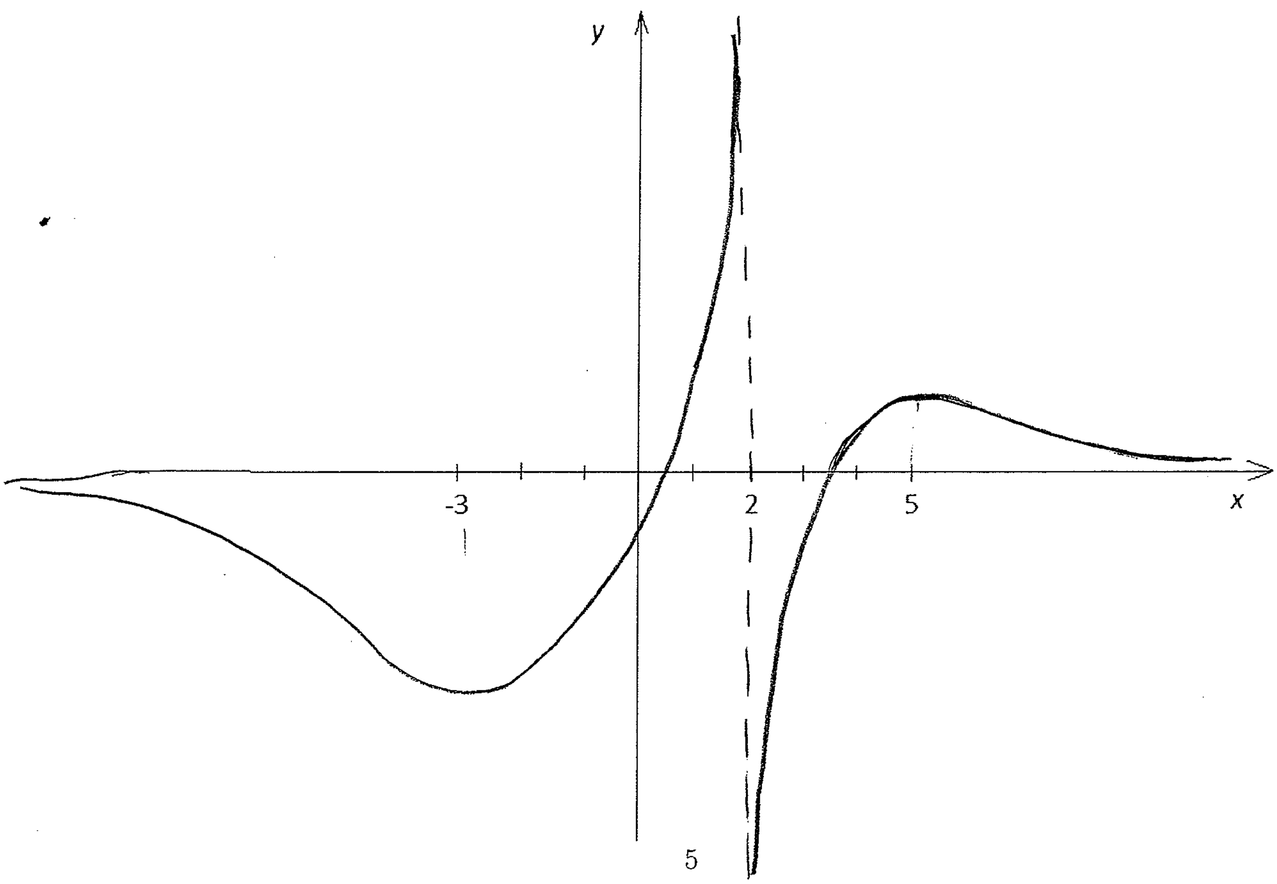
2 (c) $\lim_{x \to 2^-} f(x) = \infty$ and $\lim_{x \to 2^+} f(x) = -\infty$.

2 (d) $f'(x) < 0$ for all $x < -3$ and $f'(x) < 0$ for all $x > 5$.

2 (e) $f'(-3) = 0 = f'(5)$

Distribute 2pts for each part

Sketch below the graph of a function that *clearly* satisfies all these conditions.



12. (9 pts) Consider the function $f(x) = x^3e^x$. Find all open intervals where f is increasing.

$f'(x) = 3x^2e^x + x^3e^x = x^2(3+x)e^x$ 3pts

$f'(x) = 0 \rightsquigarrow x = 0, x = -3$ 2pts

$-3 \quad 0$ 2pts

$- \quad + \quad +$

1pt 2

f is increasing on the interval(s): $(-3, \infty)$

13. (8 pts) The second derivative of a function $f(x)$ has been computed to be $f''(x) = x^2 - 2$. Find all open intervals where f is concave up.

$f''(x) = 0 \rightsquigarrow x = \pm\sqrt{2}$ 3 pts

$-\sqrt{2} \quad \sqrt{2}$ 3pts

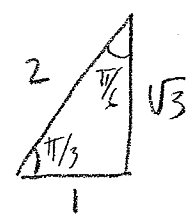
$+ \quad - \quad +$

2pts

f is concave up on the interval(s): $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, \infty)$

14. (9 pts) Consider the function $f(x) = \sqrt{3}x - 2\sin(x)$, for $0 < x < 2\pi$. Find the points of relative extrema of $f(x)$.

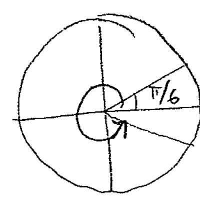
$f'(x) = \sqrt{3} - 2\cos(x)$ 2pts



$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$f'(x) = 0 \rightsquigarrow \sqrt{3} = 2\cos(x)$

$\cos x = \frac{\sqrt{3}}{2}$ 1pt



$2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}$ 2pts

includes appropriate justification

$f''(x) = 2\sin x$

$f''\left(\frac{\pi}{6}\right) = 2\sin\frac{\pi}{6} = \frac{1}{2} > 0$ $\cup$ min

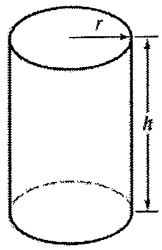
relative maximum at $x = \frac{11\pi}{6}$ 2pts

$f''\left(\frac{11\pi}{6}\right) = 2\sin\frac{11\pi}{6} = -\frac{1}{2} < 0$ $\cap$ max

relative minimum at $x = \frac{\pi}{6}$ 2pts

$\longrightarrow$ The sign of the first derivative can be used instead: $\searrow \frac{\pi}{6} \nearrow \frac{11\pi}{6} \searrow$ $\quad - \quad + \quad -$

15. (9 pts) A cylindrical can, open at the top, is to hold 510 cm^3 of liquid. Find the radius r that minimizes the amount of material needed to manufacture the can. (The surface area of the can is $S = \pi r^2 + 2\pi r h$ and the volume is $V = \pi r^2 h$.)



$\pi r^2 h = 510 \rightsquigarrow h = \frac{510}{\pi r^2}$ (1 pt)

$S = \pi r^2 + 2\cancel{\pi} r \cdot \frac{510}{\cancel{\pi} r^{\cancel{2}}} = \pi r^2 + \frac{2\cdot 510}{r}$ (2 pt)

$S'(r) = 2\pi r + 2\cdot 510 \cdot \left(-\frac{1}{r^2}\right)$ (2 pt)

(1 pt)

$S'(r) = 0 \rightsquigarrow \cancel{2}\pi r = \frac{\cancel{2}\cdot 510}{r^2}$ (1 pt)

$\pi r^3 = 510$

$r^3 = \frac{510}{\pi}$ (2 pt)

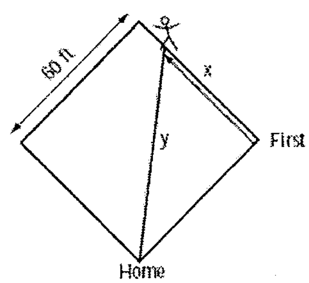
$r = \sqrt[3]{\frac{510}{\pi}}$

$-$ $+$

$\sqrt[3]{\frac{510}{\pi}}$ min

$$\boxed{r = \sqrt[3]{\frac{510}{\pi}} \text{ cm}}$$

16. (9 pts) A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is at a distance x of 20 ft from first base. At what rate is the player's distance y from home plate changing at that instant?



$x^2 + 60^2 = y^2$ (1 pt)

$\cancel{2}x \cdot \frac{dx}{dt} = \cancel{2}y \cdot \frac{dy}{dt}$ (2 pts)

$\cancel{20}\cdot 25 = \cancel{20}\sqrt{10}\,\frac{dy}{dt}$ plug in! (2 pts)

(1 pt) $x = 20$

(1 pt) $\frac{dx}{dt} = 25$

(1 pt) $y = \sqrt{20^2 + 60^2}$

$= \sqrt{400 + 3600}$

$= \sqrt{4000}$

$= 20\sqrt{10}$

$\frac{dy}{dt} = \frac{25}{\sqrt{10}}$ (1 pt)

or $\frac{25\sqrt{10}}{10} = \frac{5\sqrt{10}}{2}$

$$\boxed{\text{rate} = \frac{25}{\sqrt{10}} \text{ ft/s}}$$

17. (8 pts) Evaluate the following indefinite integral using a substitution: $\int \cos^3 x \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

3pts.

$$\int \cos^3 x \cdot \sin x dx = \int u^3(-du) = -\frac{u^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

2pts 2pts 1pt

$$\boxed{\int \cos^3 x \sin x \, dx = -\frac{\cos^4 x}{4} + C}$$

18. (9 pts) A particle moves with a velocity $v(t)$ m/s along an s-axis, where $v(t) = 1 + \frac{1}{t^2}$. Find the distance traveled by the particle during the time interval $1 \le t \le 2$.

$$\underbrace{\int_1^2 1 + \frac{1}{t^2} \, dt}_{\text{3pts for Knowing to compute the integral.}} = \int_1^2 1 + t^{-2} dt = \underbrace{t + \frac{t^{-1}}{-1}\Big/_1^2}_{4\text{pts}} = \left(t - \frac{1}{t}\right)\Big/_1^2$$

$$= \underbrace{\left(2 - \frac{1}{2}\right) - \left(1 - \frac{1}{1}\right)}_{2\text{pts}} = \frac{3}{2}$$

$$\boxed{\text{distance} = \frac{3}{2}}$$

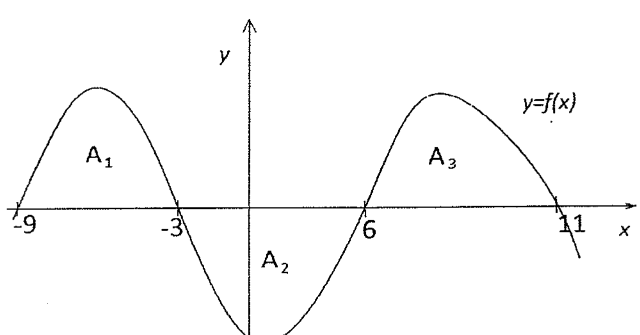
19. (5 pts) The graph of a function $f(x)$ is sketched below.

Let A_1 denote the area between $y = f(x)$ and the x-axis over the interval $[-9, -3]$.

Let A_2 denote the area between $y = f(x)$ and the x-axis over the interval $[-3, 6]$.

Let A_3 denote the area between $y = f(x)$ and the x-axis over the interval $[6, 11]$.

If $A_1 = 12$, $A_2 = 14$ and $A_3 = 15$, compute the integral $\int_{-9}^{11} f(x) \, dx$.



$12 - 14 + 15 = 13$

Give 1pt if the answer is $12 + 14 + 15$

No other partial credit.

$$\boxed{\int_{-9}^{11} f(x) \, dx = 13}$$

3 pts

20. (6 pts) Evaluate the integral $I = \int_0^{\ln 2} 5e^{3x}\, dx$.

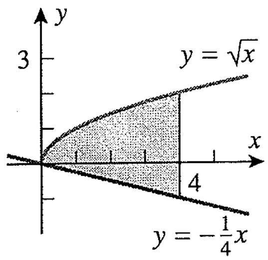
$$\underbrace{5\frac{e^{3x}}{3}\Big/_0^{\ln 2}}_{3\text{pts}} = \frac{5}{3}\cdot e^{3\ln 2} - \frac{5}{3}e^0 = \frac{5}{3}2^3 - \frac{5}{3}$$

$$= \frac{5}{3}(8-1) = \frac{5\cdot 7}{3} = \frac{35}{3}$$

$\boxed{I = \frac{35}{3}}$

(Subtract 1pt or even 2pt if the answer is not simplified!)

21. (10 pts) Find the area of the shaded region:



$$\underbrace{\int_0^4 \sqrt{x} - (-\tfrac{1}{4}x)\, dx}_{4\text{pts for setting up}} = \int_0^4 x^{1/2} + \frac{1}{4}x\, dx = \underbrace{\frac{x^{3/2}}{3/2} + \frac{1}{4}\cdot\frac{x^2}{2}\Big/_0^4}_{4\text{pts}}$$

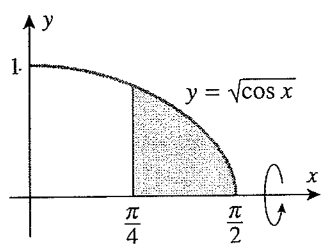
$$= \frac{2}{3}(\sqrt{x})^3 + \frac{1}{8}x^2\Big/_0^4$$

$$= \frac{2}{3}\cdot 8 + \frac{1}{8}\cdot 16 = \frac{16}{3} + 2 = \frac{22}{3}$$

2pts

$\boxed{\text{AREA} = \frac{22}{3}}$

22. (9 pts) Find the volume of the solid that results when the shaded region is revolved about the x-axis.



$$V = \int_{\pi/4}^{\pi/2} \pi(\sqrt{\cos x})^2\, dx$$

4pts for setting up

$$= \pi\int_{\pi/4}^{\pi/2} \cos(x)\, dx = \underbrace{\pi\cdot \sin(x)\Big/_{\pi/4}^{\pi/2}}_{3\text{pt}} = \pi\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)$$

$$\underbrace{= \pi\left(1 - \frac{\sqrt{2}}{2}\right)}_{2\text{pts}}$$

$\boxed{\text{VOLUME} = \pi\left(1 - \frac{\sqrt{2}}{2}\right)}$