Name: Answers

Instructor: ____________________

Time of your class: __________

UMKC Department of Mathematics and Statistics

Math 210 CALCULUS I

Common Final Examination

Saturday , December 8th 2012

INSTRUCTIONS:

1. Write your name, the name of your instructor and the time your class meets on top.
2. Only scientific calculators may be used for this examination. Graphing calculators and electronic communication devices are NOT permitted in the room.
3. Make sure that your test contains all problems indicated below on this cover page.
4. On page 1 ONLY: Multiple choice problems; full credit will be given for a correct answer and no credit for an incorrect answer; no work needs to be shown.
5. On all other pages: **All work MUST be shown to receive full credit.**
6. Unless specifically asked not to simplify, all answers need to be simplified completely.
7. Exact answers are expected, NO approximations (unless otherwise specified).
8. Write your final answer in the box provided.

Problem	Possible	Earned
1	18	
2	6	
3	8	
4	6	
5	12	
6	8	
7	8	
8	10	
9	16	
10	10	
11	6	
12	8	
13	10	
14	8	
15	10	
16	10	
17	10	
18	10	
19	8	
20	10	
21	8	
Total	200	

1. (18 pts) Determine the following limits. Choose "DNE" (Does Not Exist) as your answer only if your function has NO limit, finite or infinite. **Circle one of the indicated choices.** No work needs to be shown.

(a) (6 pts) $\lim_{x\to 3} \frac{x^2 - 6x + 9}{x^2 + 2x - 15} =$

Two methods

L'H $=$ $\lim_{x\to3} \frac{2x-6}{2x+2} = \frac{2(3)-6}{2(3)+2} = \frac{0}{8} = 0$

Type $\frac{0}{0}$

Factor

$\lim_{x\to 3} \frac{(x-3)\cancel{(x-3)}}{\cancel{(x-3)}(x+5)} = \frac{0}{8} = 0$

A. 1

B. 2

C. DNE

D. ∞

E. $-\infty$

$\boxed{\text{F. } 0}$

(b) (6 pts) $\lim_{x\to\infty} \frac{\sqrt{9x^2+10}}{x+3} =$

$\lim_{x\to\infty} \frac{\frac{\sqrt{9x^2+10}}{|x|}}{\frac{x+3}{|x|}} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{10}{x^2}}}{1+\frac{3}{x}} = \frac{\sqrt{9}}{1}$

$= 3$

Replace

$|x| = \sqrt{x^2}$

$|x| \to x$ as $x \to \infty$

$\boxed{\text{A. } 3}$

B. -3

C. DNE

D. ∞

E. $-\infty$

F. 0

(c) (6 pts) $\lim_{x\to 2^-} \frac{x+2}{x^2-4} =$

$\lim_{x\to 2^-} \frac{\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \lim_{x\to 2^-} \frac{1}{x-2} = \frac{1}{-(\text{going to zero})}$

$\overrightarrow{2}$

left of 2

always < 2

$= -\infty$

A. 1

B. -1

C. DNE

D. ∞

$\boxed{\text{E. } -\infty}$

F. 0

2. (6 pts) Consider the function

$$f(x) = \begin{cases} 6x - 2, & \text{for } x < -1 \\ kx^2, & \text{for } x \geq -1 \end{cases}$$

Find a value of the constant k, if possible, that will make the function $f(x)$ continuous everywhere.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(1)$$

$$6(-1) - 2 = k(-1)^2 = k(-1)^2$$

$$-8 = k$$

$k =$ -8

3. (8 pts) Consider the function $y = \sqrt{x^5} + \frac{1}{3}x^3 - 1$. Find the instantaneous rate of change of y with respect to x at the point with $x = 4$.

$$y = x^{5/2} + \frac{1}{3}x^3 - 1$$

$$\frac{dy}{dx} = \frac{5}{2}x^{3/2} + x^2 + 0$$

$$\left.\frac{dy}{dx}\right|_{x=4} \quad \frac{5}{2}(4)^{3/2} + (4)^2 = 36$$

rate= 36

4. (6 pts) A function $f(x)$ satisfies $f(1) = -2$ and $f'(1) = 3$. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 1$.

$$y - f(1) = f'(1)(x-1)$$

$$y - (-2) = 3(x-1)$$

$$y + 2 = 3x - 3$$

$$\underline{y = 3x - 5}$$

$y =$ 3x-5

5. (12 pts) Find the derivative of the following function; do not simplify your answer.

$$f(x) = 3\sqrt{x^2+1} + \ln(x + \cos^2 x)$$

$$\boxed{f'(x) = 3 \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x + \frac{1}{x+\cos^2 x}\left(1 + 2\cos x(-\sin x)\right)}$$

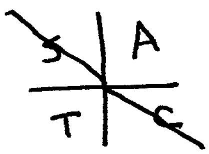
6. (8 pts) Consider the function $f(x) = e^x \cdot \sin x$. Find all the values of x with $0 \leq x \leq 2\pi$ at which the graph of f has a horizontal tangent line.

$$f'(x) = e^x \sin x + e^x \cos x = 0$$

$$e^x(\sin x + \cos x) = 0$$

$$e^x \neq 0 \qquad \sin x + \cos x = 0$$

$$\sin x = -\cos x$$



$$\tan x = -1$$

$$\boxed{x = \frac{3\pi}{4}, \frac{7\pi}{4}}$$

7. (8 pts) Let $f(x) = \dfrac{x+1}{x}$. Compute $f'(2)$ and $f''(2)$.

$$f'(x) = \frac{x - (x+1)}{x^2} = \frac{-1}{x^2} \Rightarrow f'(2) = \frac{-1}{4}$$

$$f''(x) = -\frac{0 - 2x}{x^4} = \frac{2}{x^3} \Rightarrow f''(2) = \frac{2}{8} = \frac{1}{4}$$

$$\boxed{f'(2) = -\frac{1}{4}}$$

$$\boxed{f''(2) = \frac{1}{4}}$$

8. (10 pts) If $x^3 + y^3 = 4y$, find $\dfrac{dy}{dx}$ at the point with $(x, y) = (1, 1)$.

$$3x^2 + 3y^2\frac{dy}{dx} = 4\frac{dy}{dx}$$

$$3x^2 = 4\frac{dy}{dx} \cdot 3y^2\frac{dy}{dx}$$

$$\frac{3x^2}{4-3y^2} = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{3}{4-3} = 3$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \boxed{3}$$

9. (16 pts) Find the limits and show **all** the work involved.

(a) (8 pts) $\displaystyle\lim_{x\to\pi}\frac{\sin^2 x}{(x-\pi)^2}$ Type $\frac{0}{0}$ $\overset{L'H}{=}$ $\displaystyle\lim_{x\to\pi}\frac{\cancel{2}\sin x\cos x}{\cancel{2}(x-\pi)'}$ Type $\frac{0}{0}$

$$\overset{L'H}{=} \lim_{x\to\pi}\frac{\cos^2 x - \sin^2 x}{1} = (-1)^2 = \underline{1}$$

$$\boxed{1}$$

(b) (8 pts) $\displaystyle\lim_{x\to\infty}\frac{x^2}{e^x}$ $\frac{\infty}{\infty}$ $\overset{L'H}{=}$ $\displaystyle\lim_{x\to\infty}\frac{2x}{e^x} = \frac{\infty}{\infty}$ $\overset{L'H}{=}$ $\displaystyle\lim_{x\to\infty}\frac{2}{e^x} = \frac{2}{\infty} = 0$

$$\boxed{0}$$

10. (10 pts) Consider the function $f(x) = 2x^2 + x$. Show, using the **DEFINITION** of the derivative, that $f'(x) = 4x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - [2x^2 + x]}{h}$$

$$= \lim_{h \to 0} \frac{2(\cancel{x^2} + 2xh + h^2) + \cancel{x} + h - \cancel{2x^2} - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h + 1)}{\cancel{h}} = 4x + 2(0) + 1 = \boxed{4x+1}$$ ✓

11. (6 pts) For a differentiable function $f(x)$ it is known that $f(1) = 1$ and $f'(1) = -4$. Use a local linear approximation to get the approximate value of $f(1.01)$.

$$f(1.01) \approx f(1) + f'(1)(1.01 - 1)$$

$$\approx 1 + (-4)(0.01)$$

$$\approx 1 - .04$$

$$\approx 0.96$$

$f(1.01) \simeq .96$

12. (8 pts) Find the maximum and the minimum values of the function $f(x) = 4x^3 - 3x^4$ on the interval $-2 \le x \le 2$.

$$f'(x) = 12x^2 - 12x^3 = 0$$

$$= 12x^2(1 - x) = 0$$

$$x = 0, \quad x = 1$$

$f(-2) = -80$

$f(0) = 0$

$f(1) = 1$

$f(2) = -16$

max= 1

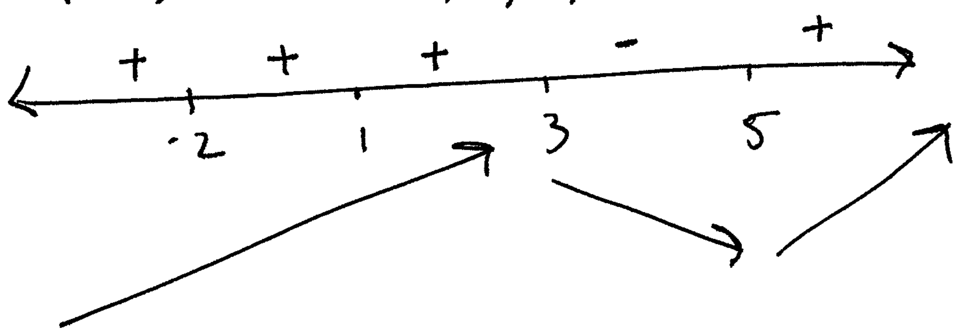
min= -80

13. (10 pts) The derivative of a function $f(x)$ has been established to be:

$$f'(x) = (x+2)^2(x-1)^2(x-3)(x-5)$$

(a) (6 pts) Find an open interval where f is decreasing.

$f'(x) = 0 \quad x = -2, 1, 3, 5$



f is decreasing on the interval: $(3, 5)$

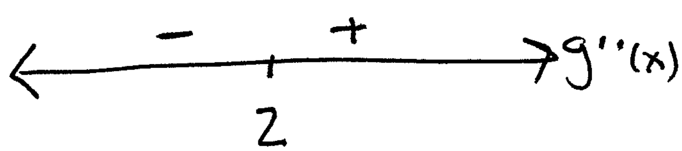
(b) (4 pts) Find all values of x for which f has a relative (local) minimum or maximum.

relative maximum at $x =$ 3
relative minimum at $x =$ 5

14. (8 pts) Determine the interval on which the function $g(x) = xe^{-x}$ is concave down.

$g'(x) = e^{-x} - xe^{-x}$

$g''(x) = -e^{-x} - [e^{-x} - xe^{-x}] = -2e^{-x} + xe^{-x} = 0$



$e^{-x}[x-2] = 0$

$x = 2$

$e^{-x} > 0$

always pos.

g is concave down on the interval: $(-\infty, 2)$

15. (10 pts) Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose area increases at a constant rate of 5 ft^2/s. How fast is the diameter of the spill increasing when the diameter of the spill is 10 ft?

given

$D = 10$

$\frac{dA}{dt} = +5 \frac{ft^2}{sec}$

want

$\frac{dD}{dt}$

know

$D = 2r$

$r = \frac{D}{2}$

$A = \pi r^2$

$\left[A = \pi\left(\frac{D}{2}\right)^2\right] \frac{d}{dt}$

$\frac{dA}{dt} = \cancel{2}\pi\left(\frac{D}{\cancel{4}}\right)\frac{dD}{dt}$

$\frac{dD}{dt} = \frac{\frac{dA}{dt} \cdot 2}{\pi D}$ $\Big|$ $\frac{dA}{dt} = 5$, $D = 10$

$\frac{dD}{dt} = \frac{5 \cdot 2}{10 \cdot \pi} = \frac{1}{\pi} \frac{ft}{sec}$

rate= $\frac{1}{\pi}$ ft/sec

16. (10 pts) A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feer of chicken wire is available for the fence?

Maximize $A = x \cdot y$

Subject to $2x + 2y = 100$ $\quad 2y = 100 - 2x$

$y = 50 - x$

so

$A(x) = x(50 - x) = 50x - x^2 \quad 0 \leq x \leq 50$

$\frac{dA}{dt} = 50 - 2x = 0$

$2(25 - x) = 0$

$x = 25$

length= 25 ft

width= 25 ft

$A(0) = 0$

$A(25) = 625$ *

$A(50) = 0$

or $A'' = -2 < 0$

c.p. is a max ✓

$4x^{-3} + \sin 3x + e^{4x}$

17. (10 pts) Evaluate the indefinite integral $\int \left(\frac{4}{x^3} + \sin 3x + e^{4x} \right) dx$

$= 4(-\frac{1}{2})x^{-2} - \frac{1}{3}\cos 3x + \frac{1}{4}e^{4x} + C$

$\boxed{-\frac{2}{x^2} - \frac{\cos 3x}{3} + \frac{e^{4x}}{4} + C}$

18. (10 pts) Evaluate the definite integral $I = \int_0^1 x(x^2+1)^3\, dx$.

$u = x^2 + 1$ $\quad$ $x=0 \rightarrow u=1$

$du = 2x\, dx$ $\quad$ $x=1 \rightarrow u=2$

$= \int_1^2 \frac{1}{2} u^3 du$

$= \frac{1}{2} \int_1^2 u^3 du$

$= \frac{1}{2}\left[\frac{1}{4}u^4 \Big|_1^2\right]$

$= \frac{1}{8}[16-1] = \frac{15}{8}$

$\boxed{I = 15/8}$

19. (8 pts) Find the area of the region between the graph of $y = \frac{2}{1+x^2}$ and the x-axis, from $x = 0$ to $x = \sqrt{3}$.

Area $= \int_0^{\sqrt{3}} \frac{2}{1+x^2} dx = 2\tan^{-1} x \Big|_0^{\sqrt{3}} =$

$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1} 0\right] = 2(\frac{\pi}{3} - 0)$

$\boxed{\text{AREA} = \frac{2\pi}{3}}$

20. (10 pts) Evaluate the integral $I = \int_{-1}^{2} f(x)\,dx$, where $f(x) = \begin{cases} 4x, & \text{for } -1 \le x \le 1 \\ 3x^2 + 1, & \text{for } 1 \le x \le 2 \end{cases}$

$$I = \int_{-1}^{1} 4x\,dx + \int_{1}^{2} (3x^2 - 1)\,dx$$

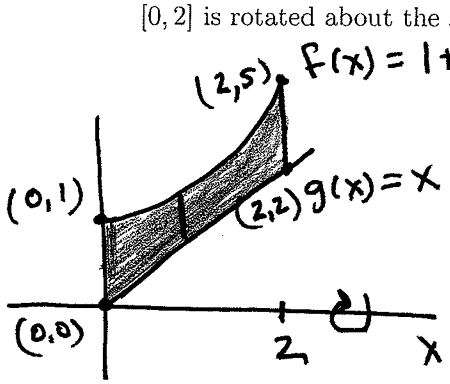
$$= \frac{4x^2}{2}\Big|_{-1}^{1} + \frac{3x^3}{3} + x\Big|_{1}^{2}$$

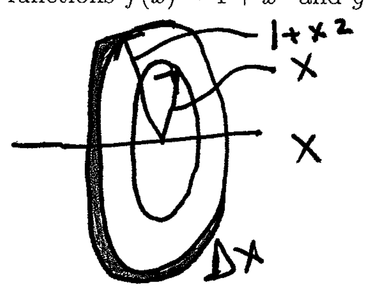
$$= 2x^2\Big|_{-1}^{1} + (x^3 + x)\Big|_{1}^{2} =$$

$$= (2-2) + (10-2) = 8$$

$I = 8$

21. (8 pts) Set up (do not evaluate) the integral for the volume of the solid that is obtained when the region between the graphs of the functions $f(x) = 1 + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is rotated about the x-axis.





Area

$\pi[(1+x^2)^2 - x^2]$

Volume

$$V = \int_0^2 \pi[(1+x^2)^2 - x^2]\,dx$$

VOLUME= $\int_0^2 \pi[(1+x^2)^2 - x^2]\,dx$